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Some Comments on Branes, G-flux, and K-theory

Gregory Moore

Department of Physics, Rutgers University,
Piscataway, New Jersey, 08855

Summary of a talk explaining three ways in which string theory and M-theory are related to the mathematics of K-theory.

1. Introduction

This talk was divided into 3 parts. In each part we describe how considerations related to D-branes and RR fields of type II string theory naturally lead to mathematics related to K-theory. Part one reviews some work done with D.-E. Diaconescu and E. Witten, and reported in [1,2]. Here we review the way K-theory theta functions are related to M-theory partition functions. Part two reports on some work in progress with G. Segal. Details will appear in a future publication [4]. Here we show how simple considerations in topological field theory lead to a picture of D-branes being classified by K-theory. Part three announced material which was done with J. Harvey, and has been described in detail in [3]. Here we explain how the K-theory of C^* algebras fits in naturally with recent developments on tachyon condensation in the presence of B-fields.

2. Partition functions: From M to K

The partition function Z_{IIA} of type IIA superstring theory on a smooth compact spin 10-manifold X should be related to the partition function Z_M of M-theory on the associated manifold $Y = X \times S^1$. Nevertheless, showing that this is the case when X is topologically complicated turns out to be a very subtle problem. This is the problem which was addressed and solved in detail in [1,2].

In brief, we consider a long-distance/weak-coupling limit. In the IIA string we take $g_{\mu\nu} = tg^0_{\mu\nu}$, where $g^0_{\mu\nu}$ is some fixed metric and we let $t \to +\infty$, $g_{string} \to 0$. On the M-theory side we take the long-distance limit (except for the M-theory circle!). In both theories the partition functions reduce to the form:

$$Z = (\text{simple} \quad \text{factors}) \times \sum_{\text{G-flux}} e^{-S(G)} \tag{1}$$

Formulating precisely both the sum and the measure in Eq.1 is tricky.

2.1. *Sum in type IIA theory*

In this case $G = G_0 + G_2 + \cdots + G_{10}$ is the sum of all fluxes. In formulating the sum there are two basic imputs one must take into account. First, even in the IIA theory, one should consider G to be a *selfdual field*: $G = *G$. Second, there is a quantization condition on the allowed fluxes. In [5,6] it is argued that the quantization condition is

$$[\frac{G}{2\pi}] = \text{ch}(x + \theta/2)(\hat{A}(TX))^{1/2} \tag{2}$$

where $[G]$ is the DeRham cohomology class of the G-field (a real differential form) while x, θ are K-theory classes. θ is a quantum shift, and *the topological sectors for RR fluxes are labelled by a K-theory class $x \in K^0(X)$*. (For a recent discussion see [7].) The definition of the partition function of a self-dual scalar field in 2 dimensions can be generalized to higher dimensions. This was done in [8] for self-dual 3-forms in the 5-brane, and generalized in [9] to the case of 10-dimensional RR fields. The procedure amounts to the quantization of a certain principally polarized abelian variety. The underlying torus is $K^0(X) \otimes R/(K^0(X)/K^0(X)_{tors})$ which turns out to have a natural metric, complex structure, and positive integral $(1,1)$ form. In this way one arrives at the type IIA K-theory theta function

$$\Theta_{IIA} = \sum e^{-KE(G)} e^{i\Phi(G)} \tag{3}$$

Here the sum runs over all cohomological values of G_0, G_2, G_4 consistent with the *existence* of a K-theory class x such that Eq. 2 can hold. The existence of such an x puts constraints on the possible cohomology class for G_4 (as described in section 2.3 below). The kinetic energy is

$$KE(G) = t^5 \parallel G_0 \parallel^2 + t^3 \parallel G_2 \parallel^2 + t \parallel G_4 \parallel^2 \tag{4}$$

and corresponds to the standard supergravity action. Here $\parallel \cdot \parallel^2$ is the standard Hodge norm in the metric $g^0_{\mu\nu}$. On the other hand, the phase is rather subtle, and

follows from the considerations of [9,1,2]. It has the structure:

$$e^{i\Phi(G)} = \Omega(x) \exp[2\pi i \int_X (-\frac{1}{15}G_2^5 + \frac{1}{6}G_2^3 G_4 + \cdots)] \tag{5}$$

The terms in the exponential are new topological phases which must be included in the 10-dimensional supergravity action in order to match properly to M-theory on a circle. The function $\Omega(x)$ takes values ± 1, and is based on a mod-two index. It can be thought of as $(-1)^{N(x)}$ where $N(x)$ is the number of Ramond fermion zeromodes on a IIB brane with K-theory charge x. One of the most striking aspects of $\Omega(x)$ is that there is no local formula for the mod two index[10] ! This is related to the well-known difficulties in formulating the action for a chiral field, and constitutes a significant departure from standard Lagrangian formulations of field theory.

2.2. *M-Theory Partition Function*

Now we define precisely the M-theory partition function on a compact smooth spin 11-manifold Y in the long distance limit. As with the RR partition function, there is a subtle quantization condition and a phase. Both were analyzed by Witten a few years ago in [11]. The quantization condition depends on a *cohomology* class $a \in H^4(Y, Z)$, and is $[\frac{G(a)}{2\pi}] = a - \lambda/2$ where λ is the characteristic class for the spin structure on Y. It satisfies $2\lambda = p_1(TY)$. The phase is, roughly speaking $\Omega_M(C) = \exp[2\pi i \int_Y (CGG + CX_8)]$, where X_8 is the famous R^4 correction to 11-dimensional supergravity [12,13]. However, if $a \neq 0$ then C is not globally defined, and one must formulate this phase carefully. One approach (adopted in [11]) is to choose a bounding 12-manifold Z, $\partial Z = Y$ on which G extends, and set $\Omega_M(C) = \exp[2\pi i \int_Z (\alpha G^3 + \beta G X_8)]$, where α, β are certain numerical coefficients. (That such extensions even exist is a nontrivial result of Stong.) While this is perfectly correct, it can be difficult to work with. Moreover, it is not manifestly well-defined because there can be different choices for Z. The best way to understand why the phase is well-defined, as pointed out by Witten in [11], is to give the formulation of the phase in terms of E_8 gauge theory in 12 dimensions.

Here we will give a slightly different (but equivalent) definition of the M-theory phase based on work in progress with Emanuel Diaconescu [14]. An M-theory 3-form "C-field" can be defined to be a quadruple: (V, A, G, c) where $V \to X$ is an E_8 vector bundle and A is a connection on V. $G \in \Omega^4(X, R)$ is a real differential form, and $c \in \Omega^3(X, R)/\Omega_Z^3(X)$, where $\Omega_Z^3(X)$ are 3-forms with integral periods (they are necessarily closed). These data must satisfy

$$\frac{G}{2\pi} = \frac{1}{60}\text{Tr}_{248}\frac{F^2}{8\pi^2} + \frac{1}{32\pi^2}\text{Tr}R^2 + d\frac{c}{2\pi} \tag{6}$$

and are subject to an equivalence relation $(V_1, A_1, G_1, c_1) \sim (V_2, A_2, G_2, c_2)$ if $G_2 = G_1$, and $CS(A_1, A_2) = c_2 - c_1$, where CS is the Chern-Simons form associated to the above characteristic class. It should be stressed that this definition is equivalent to the statement that the M-theory 3-form is a Cheeger-Simons differential character

$C \in \widehat{H}^3(Y, U(1))$. The proof of this fact relies on the remarkable result that E_8 bundles in less than 15 dimensions are classified by a single characteristic class $a \in H^4(Y, Z)$ [15]. Let us denote the corresponding bundle by $V(a)$.

In terms of these data the M-theory phase is given by

$$\Omega_M(C) = \exp\left[2\pi i \left(\frac{\eta(D_{V(a)}) + h(D_{V(a)})}{4} + \frac{\eta(D_{RS}) + h(D_{RS})}{8}\right)\right] \cdot \omega(c) \quad (7)$$

Here D_V is the Dirac operator coupled to the connection A on the bundle $V(a)$, D_{RS} is the Rarita-Schwinger operator, $h(D) =$ number of zeromodes of the operator D on Y, and $\eta(D)$ is the eta invariant of Atiyah-Patodi-Singer. The phase $\omega(c)$ is given by

$$\omega(c) = \exp\left[2\pi i \int_Y \left(c(\bar{G}^2 + X_8) + cdc\bar{G} + \frac{1}{3}c(dc)^2\right)\right] \quad (8)$$

where $\bar{G} = \frac{1}{60}\text{Tr}_{248}\frac{F^2}{8\pi^2} + \frac{1}{32\pi^2}\text{Tr}R^2$. The generalization of this formula to 11-manifolds with boundary is not trivial, and will be reported in [14]. [a]

2.3. *Equality of the partition functions*

In the case when $Y = X \times S^1$, with the supersymmetric spin structure on the circle S^1, and a "C-field" pulled back from X we can compare the partition functions at leading order, $\mathcal{O}(e^{-t})$ in the large distance expansion. Such contributions come solely from G_4. In [1,2] it is shown that both partition functions reduce to determinants times the theta function

$$\Theta = \sum e^{-\|G(a)\|^2}(-1)^{f(a)} \quad (9)$$

where the sum is over cohomology classes $a \in H^4(X, Z)$ such that $Sq^3(a) = 0$, where Sq^3 is the Steenrod squaring operation. On the IIA side, this is precisely the condition for the existence of a K-theory class x with $\text{ch}(x) = a + \cdots$. On the M-theory side, this arises because the subtle M-theory phase is sensitive to torsion information in the cohomology class a of the M-theory 4-form. The sum over the torsion classes projects onto a's such that $Sq^3(a) = 0$. It is in this somewhat indirect way that the classification of IIA RR fluxes via K-theory turns out to be compatible with the classification of M-theory 4-flux by cohomology. The $Z/2Z$-valued function $f(a)$ is a mod-two index for the Dirac operator coupled to an E_8 connection in 10-dimensions, and again follows, nontrivially, from the definition of the phase in both M-theory and in IIA theory.

2.4. *Further results*

The basic computation sketched above can be extended in many directions. First, as shown in [1,2] the result extends to the next subleading order in the large

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[a]The formula for $\omega(c)$ reported in the talk at Strings2000 omitted the second and third terms above.

t expansion at order e^{-t^3}. This involves summing over nontrivial circle bundles $Y \to X$ on the M-theory side, and correspondingly over nontrivial configurations for G_2 as well as for G_4 on the IIA side. There are many directions for further work including computation of one-loop determinants, extension to the type I setting, and inclusion of topologically nontrivial B-fields. [b]Work in progress with E. Diaconescu [14] is extending the result to include "instanton" amplitudes where membranes and fivebranes are inserted in the partition function.

Among the many open problems in the subject two are outstanding, but probably very difficult. First, E_8 gauge theory has thus far proved to be a mathematical convenience. Whether or not there is a deeper physical significance to the presence of 11 and 12-dimensional E_8 gauge theories is a fascinating open issue. Second, it is clear that the relation of K-theory to RR charges should have a profound generalization to include fundamental string and NS 5-brane charges. One focal point for finding this generalization is the issue of how the relation of RR charges and K-theory can be compatible with the S-duality of IIB string theory. (This was, in fact, the central motivating question behind [1,2].) For some provocative recent results on this subject see [16].

3. Sewing constraints and D-branes

The previous section focused on a long-distance approach to the emergence of K-theory. It is of interest to understand how the relevance of K-theory can be seen from a more "microscopic" or fundamental approach, for example, from considerations of conformal field theory, or of string field theory. Such considerations lead one directly to think about the K-theory of algebras, a subject that was very much "in the air" at the Strings2000 meeting. In this section we report on one way of understanding the connection between K-theory and D-branes based on simple considerations of sewing and topological field theory. This is based on on-going work with Graeme Segal. We recommend Segal's Stanford lectures [17] as useful background.

The axiomatic formulation of open and closed strings was considered some time ago by P. Horava [18]. Nevertheless, we think it is interesting to reformulate it in the light of the connection between K-theory and D-branes. After submission of this manuscript to the editors, but before putting it on hep-th, a paper appeared with some overlapping results for this section [19].

Given a closed string background one can ask: "What are the possible D-branes?" Similarly, given a closed conformal field theory $\mathcal{C}$ one can ask: "What are the possible associated boundary conformal field theories?" These questions are too hard to answer at present. Nevertheless, it turns out that if one replaces a conformal field theory by a 2d topological field theory then the question is solvable, yet not entirely trivial.

Recall the ancient folktheorem that 2D topological field theories are in 1-1 cor-

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[b]The considerations of section 2.1 extend straightforwardly to the case of flat B-fields $B \in H^2(X, U(1))$, if $[H] \in H^3(X, Z)$ is torsion. They reproduce rather nicely the couplings of the RR sector to the B-field.

respondence with commutative Frobenius algebras: In/out circles map to in/out Hilbert spaces, and an oriented surface maps to a linear operator between in and out spaces. The multiplication in the Frobenius algebra is defined by the pants diagram, the trace and unit are defined by the operator corresponding to the disk, with opposite orientations. The axioms of a commutative associative algebra are equivalent to the consistency of sewing arbitrary surfaces from these three components. Let us now allow both open and closed strings. Then, the 2 dimensional surfaces have two kinds of boundaries. There are in- and out-going intervals corresponding to the in- and out-going open strings, and there are the "free boundaries" corresponding to the ends of the open string "moving along a D-brane." Free boundaries carry boundary condition labels $a, b, \ldots$ which should be thought of as objects in a linear category [17]. The open string sectors $\mathcal{O}_{ab}$ are the morphisms between the objects a and b in the linear category. Our question: "Given a closed 2d topological field theory, what are the possible boundary conditions?" should be asked as two questions: First, "What are the algebraic conditions that encode consistency of open and closed string sewing?" and second, "What linear categories are consistent with these conditions?" It is useful to focus first on the case of a single boundary condition on both ends of the open string, that is, to let $\mathcal{O} = \mathcal{O}_{aa}$. Then we have

Theorem 1: To give an open and closed topological field theory is to give
   1. A commutative Frobenius algebra $(\mathcal{C}, \theta_{\mathcal{C}}, 1_{\mathcal{C}})$.
   2. A not necessarily commutative Frobenius algebra $(\mathcal{O}, \theta_{\mathcal{O}}, 1_{\mathcal{O}})$.
   3. A homomorphism $\iota_* : \mathcal{C} \to Z(\mathcal{O})$, where $Z(\mathcal{O})$ is the center of $\mathcal{O}$, such that $\iota_*(1_{\mathcal{O}}) = 1_{\mathcal{C}}$, and such that:

$$\pi = \iota_* \iota^* \tag{10}$$

Here ι^* is the adjoint to ι_*, defined by $\theta_{\mathcal{O}}(\psi \iota_*(\phi)) = \theta_{\mathcal{C}}(\iota^*(\psi)\phi)$, while $\pi : \mathcal{O} \to \mathcal{O}$ is an operator defined by the double-twist diagram.

   The condition Eq.10 is closely related to the "Cardy condition" of boundary conformal field theory. The operator π is defined using only the open string data $(\mathcal{O}, \theta_{\mathcal{O}}, 1_{\mathcal{O}})$. Indeed, if ψ_μ is a basis for $\mathcal{O}$ and ψ^μ is a dual basis relative to $\theta_{\mathcal{O}}$ then $\pi(\psi) = \sum_\mu \psi_\mu \psi \psi^\mu$. In pictures, Eq. 10 follows because the double twist diagram can be viewed both as an open string diagram, and also as a closed string channel diagram. Indeed ι^* is the open to closed string transition, while ι_* is the closed to open transition.

   The claim of the theorem is that in the case of open and closed strings with a single type of boundary condition on all free boundaries, the above conditions are equivalent to the consistency conditions for sewing. This statement is not trivial to prove. The conditions are roughly the same as those found in [20] but are not precisely the same.

   Let us now consider the solutions to the above conditions. In general this is a difficult problem, but if the "fusion rules" (i.e. the regular representation matrices) of $\mathcal{C}$ are diagonalizable then we can classify the $\mathcal{O}$'s. When the fusion rules

are diagonalizable we say $\mathcal{C}$ is "semisimple." In this case we may introduce basic idempotents such that $\epsilon_i \epsilon_j = \delta_{ij}\epsilon_i$ and $\mathcal{C} = \oplus_i C \cdot \epsilon_i$. Indeed, given any basis ϕ_μ for $\mathcal{C}$ one can diagonalize the fusion rules with a matrix S^i_μ. Letting $\mu = 0$ correspond to the identity element we can write: $\epsilon_i = \sum_\mu S^i_0 (S^{-1})^\mu_i \phi_\mu$.

Theorem 2: If $\mathcal{C}$ is semisimple then $\mathcal{O} = \mathrm{End}_\mathcal{C}(M)$ for M a finitely generated projective $\mathcal{C}$-module.

The theorem states that the possible $\mathcal{O}$'s are simply sections of a vector bundle over the "spacetime" $\mathrm{Spec}(\mathcal{C})$. One should think of ϵ_i as corresponding to the spacetime points in $\mathrm{Spec}(\mathcal{C})$. Recall that in the Gelfand-Naimark theorem one defines $\mathrm{Spec}(\mathcal{C})$ as the space of characters of the algebra $\mathcal{C}$. Indeed, the character corresponding to ϵ_i is $\chi(\phi) = \theta_\mathcal{C}(\epsilon_i \phi)/\theta_i$ where $\theta_i = \theta_\mathcal{C}(\epsilon_i)$. These traces are part of the invariants of a Frobenius algebra. Then $\mathcal{O} = \oplus_i \mathrm{End}(W_i)$ for some collection of finite dimensional vector spaces W_i.

In this formalism one can easily work out the formula for the "boundary state," defined by $B = \iota^*(1_\mathcal{O})$. This object is an element of $\mathcal{C}$ and has the property that insertions of n factors of B in $\theta_\mathcal{C}(B^n \cdots)$ corresponds geometrically to adding n holes to the surface. In formulae $B = \sum_i \dim(W_i)\epsilon_i/\sqrt{\theta_i}$, The squareroot in this formula is significant and is related to the standard fact that the closed string coupling is the square of the open string coupling. Moreover, if we consider a *family* of Frobenius algebras, then, if the subfamily of semisimple algebras has nontrivial fundamental group, transport around nontrivial loops can result in monodromy such that $\epsilon_i \to \epsilon_{\sigma(i)}$ where σ is a permutation, while $\sqrt{\theta_i} \to \pm\sqrt{\theta_{\sigma(i)}}$. Because of this one should generalize vector spaces to virtual vector spaces above to allow for negative integers in $\dim(W_i)$.

Now, let us consider multiple boundary conditions. The open string spaces $\mathcal{O}_{ab}$ for distinct boundary conditions a, b is a bimodule for $\mathcal{O}_{aa} \otimes \mathcal{O}_{bb}$. The Cardy condition generalizes in the obvious way $\pi^b_a = \iota_a \iota^b$, where ι^b is the open to closed transition for boundary conditions of type bb, and ι_a is the closed to open transition for boundary conditions of type aa. This condition can be shown to imply that $\mathcal{O}_{ab} = \mathrm{Hom}(W_a, W_b)$ where W_a, W_b are projective modules for $\mathcal{C}$. The conclusion then is that *the linear category classifying the boundary conditions for the closed 2d topological field theory is the K-theory $K_0(\mathcal{C})$ for the commutative closed string Frobenius algebra.*

Many interesting examples of the above theorems applied to families of Frobenius algebras can be given. Moreover, the discussion can be generalized to "orbifolds" that is, to the equivariant case. Analogs of theorems 1 and 2 above can be stated, at least in the semisimple case. It is possible to introduce a "B-field" even in this topological setting. The "untwisted sector" of the theory defines an ordinary commutative Frobenius algebra $\mathcal{C}_1$, and $\mathrm{Spec}(\mathcal{C}_1)$ will be a finite G-space X, where G is the orbifold group. The B-field is valued in G-equivariant cohomology $H^2_G(X; U(1))$ and has a corresponding fieldstrength $h \in H^3_G(X; Z)$. The linear category is then

given by the K-theory $K_{G,h}(X)$ of G-equivariant, twisted bundles of algebras over the "spacetime" X, in accord with previous works. Details will be described in [4].

The above results might have some bearing on the subject of topological open strings and D-branes in Calabi-Yau manifolds. For recent work on this subject see [21,22,23].

4. Noncommutative Tachyons and K-theory

Part three of the talk announced results that have since been published in [3]. We will therefore be very brief here.

There has been much recent progress on understanding tachyon condensation by using the technology of noncommutative field theory [24,25,26,27,28]. Consider the bosonic or the type II string. The general picture is to consider spacetime to be a product of a commutative and a noncommutative manifold $X_c \times X_{nc}$. A field on a noncommutative manifold is equivalent to an operator on Hilbert space. Therefore, if an unstable D25, or D9 brane wraps this spacetime then its tachyon field will be described by a field on X_c, valued in operators on Hilbert space. At large noncommutativity parameter the equations of motion for the tachyon field say that it is a projection operator (in the bosonic string) or a partial isometry (in the type II string). This fits in perfectly with the K-theory classification of D-branes. Indeed, the isomorphism classes of complex rank n vector bundles on X_c are in one-one correspondence with homotopy classes $[X_c, BU(n)]$ where $BU(n)$ is the space of rank n projection operators on Hilbert space. Similarly, the Atiyah-Janich model for K-theory shows that we can identify $K^0(X_c)$ with the homotopy classes $[X, \mathcal{F}]$ where $\mathcal{F}$ is the space of Fredholm operators on Hilbert space. By polar decomposition one can restrict to the space of partial isometries.

In the type II string a very special class of partial isometries can be written corresponding to the "noncommutative Atiyah-Bott-Shapiro construction." One takes $[x^i, x^j] = -i\theta^{ij}$ with θ of maximal rank and $i = 1, \ldots, 2p$. Letting Γ_i be chiral gamma matrices of rank 2^{p-1} we can form the partial isometry in the polar decomposition of $\Gamma_i x^i$. This is the tachyon field corresponding to condensing a D9 brane to a D(9-2p) brane transverse to a noncommutative plane of dimension 2p. The fact that the D-brane charge is unchanged by turning on a B-field is equivalent to the index theorem identifying topological and analytical indices. For further details, and further developments of these ideas, see [3], and references therein.

Recently, in collaboration with E. Martinec we have generalized these ideas to include D-branes on orbifolds. The algebra of functions on X_{nc} is replaced by a crossed-product algebra. Fractional branes and discrete torsion are very naturally and easily incorporated into the formalism. In addition, the formalism gives an interesting perspective on the formulation of the theory of D-branes in asymmetric orbifolds. Details will appear in [29].

Acknowledgments

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